Complementarity of phase transition and stochastic resonance in spatially restricted systems

Artjom Vargunin, Teet Örd, and Risto Tammelo*
Institute of Theoretical Physics, University of Tartu, 4 Tähe Street, 51010 Tartu, Estonia
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We study the linear response in a soft potential, the Landau-type temperature dependence of which is responsible for the monostable-to-bistable transformation. The system can be considered as a model of a sample with internal noise, where a second-order phase transition takes place in the bulk limit. The intensity of noise influencing the order parameter is supposed to be a function of the volume. We demonstrate that the anomaly of susceptibility at the phase transition point described by the Landau phase transition theory transforms into the maximal response caused by stochastic resonance if the volume of the system decreases. The phenomenon can be treated also as an increase in the diffuseness of a phase transition with lowering of the critical temperature. We suggest that it is this crossover that contributes to the mechanism of dielectric peculiarities in ceramic and relaxor ferroelectrics.

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I. INTRODUCTION

Bistable models have been an extremely fruitful concept which allows one to explain various effects in physics, chemistry, biology, etc. The models of this type describe, at least in the thermodynamic limit, cooperative phenomena, such as magnetic or structural orderings; e.g., see Refs. [1,2]. At the same time, bistability is a necessary attribute for the phenomenon of stochastic resonance [3–8], which indicates the maximal absorption of the energy of an external temporally regular field due to ordered (correlated) dynamics. Recently, an interrelationship between dynamic phase transitions [9,10] and stochastic resonance has been established in spatially extended bistable systems [11].

The dynamical susceptibility of the system which undergoes a second-order phase transition in the thermodynamic limit exhibits the well-known anomalous behavior in the vicinity of the phase transition temperature [12,13]. This resonantlike dependence has a different nature compared to stochastic resonance, although the latter is also characterized by maximum susceptibility to temperature. However, one can suppose that there is a possibility for the transformation from the regime of the stochastic resonance in a small sample to the enhanced response at the phase transition point in a macroscopically large sample. This hypothesis stems from the assumption that the role of the internal noise increases when the system's dimension decreases.

In the present communication, we study the noise-induced resonance of the response to the temporally periodic field in a soft potential of a temperature-dependent shape. Since the intensity of noise is considered to be a function of the volume, one can control the response by the volume of the model system and observe, at least approximately, its asymptotical behavior when approaching the macroscopic situation.

II. SOFT POTENTIAL WITH VOLUME-DEPENDENT NOISE INFLUENCING THE ORDER PARAMETER

We start with the one-dimensional equation of overdamped motion

$$\frac{dx(t)}{dt} = -\frac{\partial U(x;T)}{\partial x} + A(t) + \sqrt{\frac{T}{V}}\xi(t), \tag{1}$$

where $\xi(t)$ is the zero mean Gaussian white noise with the correlation function $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')$; here, T is the temperature, V is the volume of the sample, $A(t)=A_0\cos(\Omega t)$ is a periodic force, and a temperature-dependent soft potential is taken in the Landau form [12]

$$U(x;T) = \frac{1}{2}a(T)x^2 + \frac{1}{4}x^4, \tag{2}$$

with $a(T)=\alpha(T-T_c)$, where the constant $\alpha>0$. The potential $U(x;T)$ is bistable if $T<T_c$ and monostable if $T>T_c$. In the static case ($\Omega=0$), the statistic equilibrium of the system (1) is described by the partition function

$$Z = \int_{-\infty}^{\infty} \exp\left\{-\frac{V}{T}[U(x;T) - A_0 x]\right\} dx. \tag{3}$$

The latter coincides with the partition function of a small particle of a system undergoing in the bulk limit a second-order phase transition described by the order parameter x. Here the smallness of the particle implies that its dimensions are smaller than the correlation length of the order parameter fluctuations [14]. Note also that the free-energy density determined as $\Phi=-TV^{-1}\ln Z$ reduces in the limit $V\to\infty$ to the minimal (equilibrium) value of the homogeneous Landau free energy [12]. Furthermore, the infinite-volume limit in the equation of motion (1) leaves us with the Landau-Khalatnikov equation describing the relaxation of the order parameter to its equilibrium position. On the other hand, it seems to be legitimate that as the dimension decreases, the fluctuations of the order parameter will increase, so the Langevin equation (1) describes, at least qualitatively, correctly the evolution of the order parameter x in the samples with bulk second-order phase transitions.

III. DYNAMIC SUSCEPTIBILITY

According to the scheme developed in Ref. [4], the linear dynamic susceptibility of the system under consideration is expressed in the asymptotic time limit as

*risto.tammelo@ut.ee

$$\chi(T,\Omega) = \frac{V}{T}\left[\left(\frac{g_1\lambda_1^2}{\lambda_1^2+\Omega^2} + \frac{g_3\lambda_3^2}{\lambda_3^2+\Omega^2}\right) - i\Omega\left(\frac{g_1\lambda_1}{\lambda_1^2+\Omega^2} + \frac{g_3\lambda_3}{\lambda_3^2+\Omega^2}\right)\right], \quad (4)$$

where

$$g_1 = \langle x^2\rangle_{st} - g_3, \quad (5)$$

$$g_3 = \frac{[\lambda_1 + \alpha(T_c - T)]\langle x^2\rangle_{st} - \langle x^4\rangle_{st}}{\lambda_1 - \lambda_3}. \quad (6)$$

Here $\langle\cdots\rangle_{st} = \int_{-\infty}^{\infty}\cdots P_{st}(x)dx$, where $P_{st}(x)$ is the stationary probability distribution of the nonperturbed system, and $\lambda_{1,3}$ are the first and third eigenvalues of the nonperturbed Fokker-Planck operator associated with the Langevin equation (1):

$$\hat{L}_{FP}(x) = \frac{\partial}{\partial x}\frac{\partial U(x;T)}{\partial x} + \frac{T}{V}\frac{\partial^2}{\partial x^2}. \quad (7)$$

Note also that only odd eigenvalues contribute to Eq. (4) in accordance with Ref. [15]. We calculate the eigenvalues $\lambda_{1,3}$ numerically, solving the corresponding Schrödinger equation [16] by means of the symplectic method; see, e.g., Ref. [17]. The relevant Schrödinger operator has the form

$$\hat{L}_S(x) = \frac{T}{V}\frac{\partial^2}{\partial x^2} - U_S(x;T), \quad (8)$$

with the potential

$$U_S(x;T) = \frac{V}{4T}\left(\frac{\partial U(x;T)}{\partial x}\right)^2 - \frac{1}{2}\frac{\partial^2 U(x;T)}{\partial x^2}. \quad (9)$$

In Eq. (4), the terms proportional to the coefficient g_1 describe the contribution from the interwell dynamics and the terms proportional to the coefficient g_3 describe the contribution from the intrawell dynamics to the linear response of the system in the bistable regime [4]. As revealed by the analysis of the numerically obtained eigenvalues $\lambda_{1,3}$, a sufficiently large time-scale separation between the interwell hopping and intrawell motion holds for a whole range of parameters under consideration. This is the only substantial restriction for the present scheme. In a rigid bistable potential, the interwell processes control the phenomenon of stochastic resonance, while the intrawell processes manifest themselves mainly in the weak-noise limit behavior of susceptibility [3,4]. Thus, one can expect the intrawell dynamics to be expressed in our model mainly in the low-temperature or large-volume region.

IV. CROSSOVER BETWEEN STOCHASTIC RESONANCE AND THE SECOND-ORDER PHASE TRANSITION

The dependence of susceptibility on temperature for various values of the volume and frequency is displayed in Fig. 1. In these figures the curves are calculated by means of Eq. (4). According to the Landau theory, in the bulk limit ($V\to\infty$, formally) the susceptibility has an anomaly at the phase transition point $T=T_c$, and its temperature and frequency dependences are given by the following relations [13]:

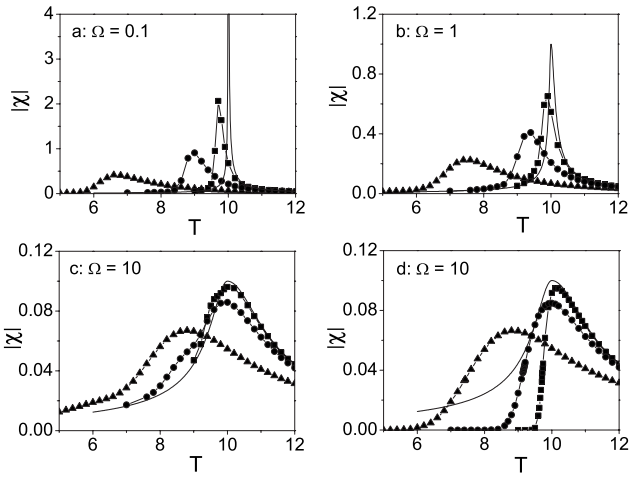

FIG. 1. The plots of the modulus of susceptibility vs temperature at frequencies (a) $\Omega=0.1$, (b) $\Omega=1$, (c),(d) $\Omega=10$ for various volumes $V=0.1$ (triangles), $V=1$ (circles), $V=10$ (squares), and the bulk limit (solid lines without symbols). Here we use $T_c=10$ and $\alpha=10$. In (d) only interwell dynamics is taken into account. (Note that all the quantities depicted in all of the figures are given in relative units; i.e., all the quantities occurring in the figures are dimensionless.)

$$|\chi(T,\Omega)| = [4\alpha^2(T_c - T)^2 + \Omega^2]^{-1/2}, \quad T < T_c, \quad (10)$$

$$|\chi(T,\Omega)| = [\alpha^2(T - T_c)^2 + \Omega^2]^{-1/2}, \quad T > T_c. \quad (11)$$

In fact, these relations follow from Eq. (4) in the macroscopic limit. As one can see, the resonant maximum of $|\chi|$ shifts to higher temperatures if the volume increases, approaching asymptotically the response anomaly at the phase transition point T_c in the infinite-volume limit. Such behavior is caused by the competition between the interwell and intrawell processes, while for smaller volumes (higher noise intensity) the interwell dynamics is the dominating one, but for larger volumes (lower noise intensity) the intrawell relaxation becomes more important in the bistable region for sufficiently high temperatures. This tendency is also observable from the comparison of cases (c) and (d) in Fig. 1. An increase in the frequency will lead to a lowering and broadening of the maxima of susceptibility together with a rise in the resonant temperature.

Thus, there is a crossover from the stochastic resonance in a small sample to the usual second-order phase transition in a macroscopic system. These anomalous phenomena, both characterized by enhanced sensitivity to the external field, are complementary to each other in the present model.

Let us now consider some experimental data. Stochastic resonance with an external source of noise was experimentally observed in ferroelectrics—for example, triglycine sulfate [18,19]. However, let us look at the ceramic [20–24] and relaxor [25] ferroelectrics. In these materials the ferroelectric transition becomes increasingly diffuse with a decrease in grain size [26]. Thus we conclude that stochastic resonance

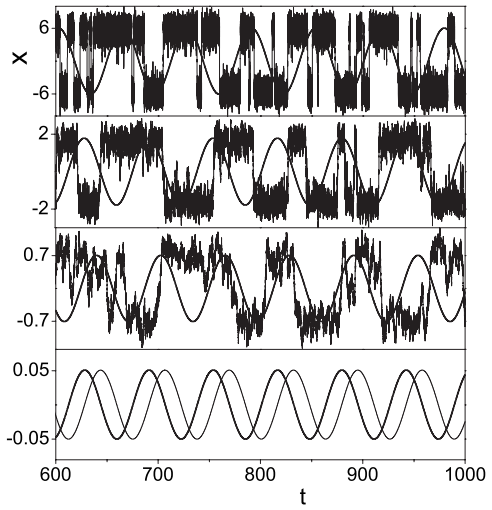

FIG. 2. The temporal realization of the order parameter x for various volumes (from top to bottom) $V=0.1$, $V=10$, $V=200$, and $V=\infty$ (bulk limit). For each volume the temperature is taken to produce the maximal response (maximal value of the modulus of the susceptibility) to the external field (thick periodic curves) with $A_0=0.005$ and $\Omega=0.1$. For convenience of the data presentation the field amplitude is appropriately renormalized. Here we use $T_c=10$ and $\alpha=10$.

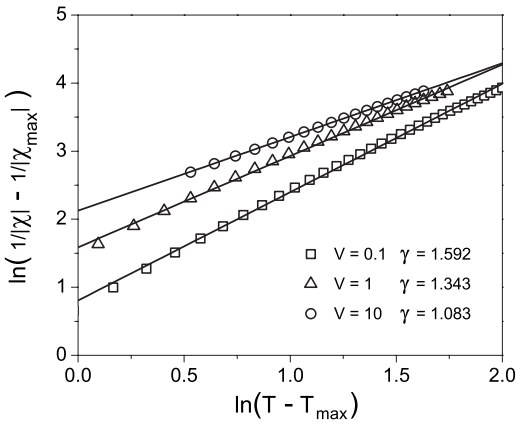

FIG. 3. The logarithmic plot of the reciprocal susceptibility $|\chi(T,\Omega)|^{-1}-|\chi(T_{\max},\Omega)|^{-1}$ measured at frequency $\Omega=1$ as a function of $T-T_{\max}$ for various volumes $V=0.1$ (squares), $V=1$ (triangles), and $V=10$ (circles) fitted to Eq. (12), solid lines. The slope determines the critical exponent γ. Here we use $T_c=10$ and $\alpha=10$.

may be of relevance for the dielectric constant measurements in these types of materials reproducing the observed diffuseness of the phase transition. Moreover, these experiments may be thought to evidence the fact that in small samples the order parameter will unavoidably become a stochastic variable.

Let us recall that our approach is based on the Landau phase transition theory. It should be emphasized that the complementarity of the phase transition and stochastic resonance discussed above is a different phenomenon compared to the crossover between the dynamic phase transition and stochastic resonance reported in Ref. [11]. The latter crossover and the dynamic phase transition itself were established in Refs. [27–29] in the bistable systems only, whereas no evidence of the existence of a dynamic phase transition above the critical temperature T_c was found, indicating the absence of this phenomenon in the monostable systems. At the same time, it was conjectured by Korniss *et al.* [27] that the critical field amplitude, which separates the regions where the dynamic transition exists and is absent, vanishes at T_c. Consequently, according to this conjecture, at the temperature T_c the dynamic phase transition cannot occur in contrast to the Landau phase transition. Thus the arguments presented in this paragraph confirm the idea that the complementarity proposed in the present contribution is a novel phenomenon. In addition, we have performed numerical simulations of our complete model; these simulations present the complementarity in terms of temporal realizations of the order parameter (see Fig. 2).

V. DIFFUSENESS OF THE PHASE TRANSITION

A diffuse phase transition is usually characterized by the following features: a broadening in the maxima of the χ-T curves, relatively large separation in temperature scale between the real and imaginary parts of the susceptibility maxima, a deviation from the Curie-Weiss law in the vicinity of the transition temperature, frequency dispersion of the susceptibility, and some others. All these signatures are realized in the present model if one considers the finite sample. The quantitative measure of the diffuseness of the phase transition can be estimated using the following equation for $T>T_{\max}$ [$T_{\max}$ corresponds to the peak in $|\chi(T,\Omega)|$] [22–24,30–32]:

$$|\chi(T,\Omega)|^{-1}-|\chi(T_{\max},\Omega)|^{-1}=C(T-T_{\max})^{\gamma}, \qquad (12)$$

where C is a constant and the critical exponent $\gamma=1$ for a classical Curie-Weiss ferroelectric, $\gamma=2$ for the system with a completely diffuse phase transition, and for systems with intermediate degrees of diffuseness $1<\gamma<2$. As follows from the nearly linear behavior in Fig. 3, diffuseness can also be determined in the present model. Moreover, the critical exponent γ seems to be frequency independent and decreases with an increase in dimension, pointing to the well-known experimental fact that a sharp phase transition can occur only in the thermodynamical limit, getting smeared in the finite system. Thus, we believe that the present scheme will be useful for understanding both the nature of diffuse phase transitions and the dielectric behavior of the polar nanodomains in relevant ferroelectric systems.

VI. SYSTEM SIZE RESONANCE

In addition to the resonance of the response driven by temperature, the existence of the system size resonance in the present scheme is illustrated in Fig. 4. As one can see, the dynamic susceptibility behaves resonantly when the dimension increases. A similar phenomenon was recently established in various ferroelectric systems both experimentally and theoretically; e.g., see Refs. [7,33–37] and references therein. However, in the present scheme the sharp maximum of the susceptibility in the low-volume region is the conse-

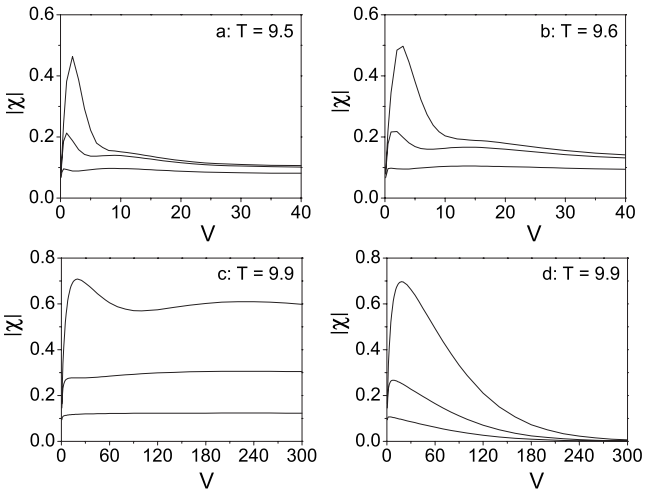

FIG. 4. The plots of the modulus of susceptibility vs the volumes at fixed temperatures (a) $T=9.5$, (b) $T=9.6$, and (c),(d) $T=9.9$ for various frequencies (from top to bottom on each figure) $\Omega=1$, $\Omega=3$, and $\Omega=8$. Here we use $T_c=10$ and $\alpha=10$. In (d) only interwell dynamics is taken into account.

quence of the size stochastic resonance [7,8,38,39] relating the latter phenomenon to the existence of the critical dimension [40] below which the ferroelectricity vanishes. In this region, the susceptibility is very sensitive to the frequency of the applied field and an effect of the size stochastic resonance is supported by its smaller values.

In the large-volume region, as one can expect, the role of the interwell dynamics and size stochastic resonance is suppressed. The main cost to the susceptibility stems from the intrawell dynamics (see Fig. 4) and the asymptotic volume behavior of the susceptibility defined in Eq. (10). However, the response of the system in the large-volume region is a nonmonotone function of the volume which can be characterized by the broad maximum. We must also emphasize that this resonance is caused solely by the intrawell processes. Thus, there exists a competition between such intrawell resonance and size stochastic resonance if the temperature approaches a critical value, once more suggesting the complementarity of the stochastic resonance and phase transition.

Also, the crucial difference between these types of resonance will become apparent if one examines the contributions of the real and imaginary parts of susceptibility (4) to the response. This analysis shows that in terms of susceptibility one can separate both temperature and volume scales into three zones. There are the intrawell vibrations, interwell transitions, and "one-well" response (these terms were taken from Ref. [41]). The left-hand side of the maximum in Fig. 4(a) belongs to the "one-well" response zone. In this region, a bistable system behaves effectively as a monostable one and the susceptibility is governed mostly by its real part. The interwell transitions manifest themselves in the right-hand side of the maximum. The resonant behavior of the imaginary part is responsible for the form of susceptibility in this region relating it to the maximal absorption of the energy of the external field. The plateau (or more precisely, a broad maximum) of the response relates to the intrawell vibrations, because for sufficiently large volumes the wells of the bistable potential do not essentially "sense" each other. In this case, the susceptibility is governed by its real part and the nonmonotonicity of the real part leads to such features as intrawell resonance. The same separation of the scale also holds for the temperature-dependent response [see, for instance, Fig. 1(a)], but only in the reverse order.

VII. CONCLUSION

We have examined the system in the Landau-type potential driven by temperature- and volume-dependent noise in an applied periodic field, establishing in terms of the dynamic response the crossover from stochastic resonance to phase transition in the bulk system controlled by the dimensions of the sample. As a result, the diffuseness of the phase transition increases when the dimension decreases. It was also demonstrated that the small samples exhibit size resonance which is purely of stochastic nature. Application to dielectric constant measurements in ceramic and relaxor ferroelectrics in both the temperature and volume scales seems to be possible.

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